

From Optimal Execution in Front of a Background Noise to Mean Field Games

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Outline



Motivation And Main Principles: Why I Believe MFG Are Perfect For Liquidity Modelling

2 How to Design a MFG For Orderbook Dynamics: Liquidity Formation

3 How to Design a MFG At a Mesoscopic Scale: Optimal Trading and Crowding

Mean Field Games As a Model For Liquidity On Financial Markets

Motivation: Recent Evolution of Financial Markets Put The Focus on Liquidity

Before the financial crisis (A "Haute Couture" Business Model)

- Products were sophisticated and highly customized
- Intermediaties (brokers, banks, etc) needed to keep large inventories (and hence hosted a lot of risk)

Since the financial crisis (mass market)

- Products are simpler and standardized
- Regulators demand for lower inventories (G20 Pittsburgh 2008)
- \Rightarrow Intermediaries turned to an flow-driven business.
- ⇒ Liquidity is an important issue for regulators, intermediaries, and their clients.

Moreover, regulators want more transparency (for less information asymmetry between intermediaries and their clients), hence they promote electronic, multilateral trading.

Basics of Financial Auction Games

Use Automated Trading in The Financial Industry



- More products can be traded electronically every year (in Europe MiFID 2 – Jan 2018– pushes fixed income products to electronic).
- Humans use a collection of automated trading algorithms and have to monitor them instead of interacting directly with auction mechanisms; the **monitoring** and **human – machine interfaces** are very important (see [Azencott et al., 2014]).
- These algorithms are explicitly parametrized by their utility function when they are used by dealing desks (Implementation Shortfall, Percentage of Volume, Volume Weighted Average Price, Smart Routing, Liquidity Seeking, etc).
- The ones used by prop traders and market makers are more based on ad hoc mixes of signals and risk control micro-strategies (cf. [L. and Neuman, 2017] for an attempt of modelling).
- Operational risk (including code architecture, online learning –like in [Laruelle et al., 2013] – and deployment mechanisms) is an important topic.

Basics of Financial Auction Games

Principles of the Auctions on Financial Markets



Bilateral Trading

Each client face one Market Maker. He asks for quotes (bid ask prices and quantities), and the market maker adjusts her prices to the level of information (toxicity) of this particular client.



Multilateral Trading

Several (anonymous) market makers and their clients trade in the same pool, all competing for liquidity.

Price is dynamically set so that buyers with low prices match seller with high prices. (real-time Walrassian

mechanism).

See [L. and Laruelle., 2018a] for details.

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Principal – Agent

See [L. and Laruelle., 2018a] for details.

Basics of Financial Auction Games

Two Main Mechanisms For Multilateral Trading





At a mesoscopic scale (~5min)

For 20 years [Almgren and Chriss, 2000], financial Mathematics developped **stochastic-control frameworks** to optimize the strategy of **one trader in front of a background noise**. Interactions with others are is reduced to

- a model for price reaction to buying or selling pressure (i.e. a market impact model);
- a martingale "innovation" rendering the aggregated behaviour of (a priori) independent other players.

Basics of Financial Auction Games

Two Main Mechanisms For Multilateral Trading



Illustration from [L., Mounjid and Rosenbaum., 2018b]

Illustration from [Bouchard et al., 2011]

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Mean Field Games For Liquidity Games

Market Liquidity Satisfies Most of the Needed Properties of the "Mean Filed" of MFG

MFG

- A continuum of players
- implementing stochastic control
- with a cost function incorporating functionals of the repartition of all players (i.e. the "mean field")
- $\rightarrow\,$ You demand anonymity, and you obtain a Nash equilibrium

See seminal papers by Lasry and Lions, and simultaneous papers by Caines, Huang and Malhamé, have a look at [Bensoussan et al., 2016] for the LQ case.

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In short: consider the trajectories of a continuum of agents, each of them described by a typical controlled stochastic processes X_t , the control minimizes a criterion involving the distribution m_t of all agents:

$$\begin{array}{rcl} dX & = & b(t, X_t, \alpha_t) \, dt + \sigma(t, X_t) \, dW, \quad X_0 = x_0 \\ \alpha_t & = & \arg\min_a I\!\!\! E \int_{s=t}^T \left\{ L(X_s, \alpha_s) + f(X_s, m_s) \right\} \, ds + g(X_T, m_T), \quad X_t = x \\ Law(X_t) & \sim & m_t \end{array}$$

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Liquidity on financial markets

- Market participants' buying and selling is expressed in terms of "I (would like to) trade up to this quantity at this price".
- The aggregation of all these intentions is a mean field, and the traded price is a function of this mean field
- Participants' costs for sure are function of this current price.
- $\rightarrow~$ Let's write this as a MFG

Mean Field Games For Liquidity Games

Some Papers are Available

This talk is based on two examples of the use of MFG to model liquidity on financial markets:

- at the high frequency time scale (orderbooks dynamics): Efficiency of the Price Formation Process in Presence of High Frequency Participants: a Mean Field Game analysis [Lachapelle et al., 2016]
- at a mesoscopic time scale (optimal trading in presence of multiple players): Mean Field Game of Controls and An Application To Trade Crowding [Cardaliaguet and L., 2017]

It is worthwhile to note that these two problems have been explored: in front of a background noise

- ▶ in [L. and Mounjid, 2016] and [L., Mounjid and Rosenbaum., 2018b] for the first one;
- and in a series of papers by Cartea and Jaimungal [Cartea et al., 2015] for the second one (that is a derivation of the initial Almgren and Chriss framework).

Other papers do explore similar mechanisms, like [Carmona et al., 2013] and [Jaimungal et al., 2015] or [Firoozi and Caines, 2016] (the two latters are close to Cardaliaguet-L.).

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A Mean Field Game Model for One Queue of the Orderbook

The Setup

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- Sellers only,
- one agent *i* arrives in "the game" at *t* according to a Poisson process *N* of intensity λ,
- it compares the value to wait in the queue (y(x), where x is the size of the queue) to zero to choose to wait in the queue (when u(x) > 0) or not, its decision is δⁱ
- the queue is consumed by a Poisson process M^{μ(x)} of intensity μ(x),
- in case of transaction, a "pro-rata" scheme is used ("equivalent" to infinitesimal possibility to modify orders): q/x of the order is part of it; can be relaxed for FIFO.



A Mean Field Game Model for One Queue of the Orderbook

Dynamics, Controls and Cost Functions

The Mean Field is the size of the queue (it is a forward process):

$$dx_t = q \left(dN_t^j \delta^j - dM_t^{\mu(x_t)} \right)$$
, remark: $j = N$, I could have written $dN_t \delta^N$.

The Value function the *i*th agent wants to minimize is driven by the following running cost

$$\begin{split} dJ(x_t) &= \left[\frac{q}{x_t} P(x_t) + (1 - \frac{q}{x_t}) J(x_t - q)\right] dM_t^{\mu(x)} - cq \, dt. \\ u(x) &:= I\!\!E \int_{t_0}^T dJ(x_t), \end{split}$$

and its control δ^i is to choose to be submitted to this cost function or to pay zero at t_0 :

$$\mathcal{U}^{i}(x) := \max_{\delta^{i} \in \{0,1\}} \delta^{i} u(x).$$

The optimal decision δ^i is the solution of the **backward** associated dynamics.

A Mean Field Game Model for One Queue of the Orderbook

Solution for a specific form of $\mu(x)$



At queue sizes x^* such that

 $x^* = \mu(x^*)P(x^*)/c$

the sign of *u* changes. Moreover, for the specific case

 $\mu(\mathbf{x}) = \mu_1 \delta_{\mathbf{x} < S} + \mu_2 \delta_{\mathbf{x} \ge S}$

There is a point strictly before *S* where *u* switches from negative to positive. It means that **participants anticipate service improvement**.

This chart has to be compared to the left Panel of Slide 30, Peter Tankov's talk (yesterday) on *Mean field games of optimal stopping: a relaxed control approach.*

A MFG of Two Interacting Queues

The decision-taking process will follow this mechanism:



- consuming liquidity allows to obtain quantity immediately but at an impacted price, with respect to the liquidity available in the book,
- each time a market participant has to take a buy or sell decision, he tries to anticipate the "*long term*" value for him to be liquidity provider or liquidity consumer,
- each market participant can use a SOR (Smart Order Router [Foucault and Menkveld, 2008]) for this sophisticated valuation, otherwise he will just consume liquidity.

A MFG of Two Interacting Queues

Model details

- Orders arrive at Poisson rate $\Lambda = \lambda + \lambda^{-}$
- strategic arrivals: λ , non-optimal: λ^- (can be read as "SOR" on "non-SOR" participants)
- $(Q_a, Q_b) :=$ number of orders on ask and bid sides
- Value functions: $u(Q^a, Q^b)$ for sellers and $v(Q^a, Q^b)$ for buyers
- Matching process for any quantity Q: Qq/Q^a
- Transaction price:

$$p^{\mathrm{buy}}(Q^a) := P + rac{\delta q}{Q^a - q}, \quad p^{\mathrm{sell}}(Q^b) := P - rac{\delta q}{Q^b - q}$$

where: q is the order size, δ is the market depth, P is the fair price

• cost to maintain inventory: ca and cb

Decision process

• If $u(Q_t^a + q, Q_t^b) > p^{sell}(Q_t^b)$, it is more valuable to route the sell order to the ask queue \rightarrow Liquidity Consumer (LC) order

• If $v(Q_t^a, Q_t^b + q) < p^{buy}(Q_t^a)$, it is more valuable to route the buy order to the bid queue \rightarrow Liquidity Provider (LP) order

• Notations of the routing decisions:

$$\begin{split} &R^{\oplus}_{\text{buy}}(v, Q_t^a, Q_t^b + q) := \delta_{v(Q_t^a, Q_t^b + q) < \rho^{\text{buy}}(Q_t^a)}, \text{LP buy order} \\ &R^{\oplus}_{\text{sell}}(u, Q_t^a + q, Q_t^b) := \delta_{u(Q_t^a + q, Q_t^b) > \rho^{\text{sell}}(Q_t^b)}, \text{LP sell order} \\ &R^{\oplus}_{\text{buy}}(Q_t^a, Q_t^b) := 1 - R^{\oplus}_{\text{buy}}(Q_t^a, Q_t^b), \text{LC buy order} \\ &R^{\otimes}_{\text{sell}}(Q_t^a, Q_t^b) := 1 - R^{\oplus}_{\text{sell}}(Q_t^a, Q_t^b), \text{LC sell order is processed} \end{split}$$

The Model in Detail

The **2D** mean field is the size of the two queues (Q_l^a, Q_l^b) ; it evolves according to the forward dynamics (j, k, ℓ) are strategic ask providing, strategic ask consuming, and blind ask consuming agents):

$$dQ_t^{a} = \left(dN_t^{\lambda_{\text{sell}}(j)}, \underbrace{R_{\text{sell}}^{\oplus}(j)}_{\delta^j} - \left(dN_t^{\lambda_{\text{buy}}(k)}, \underbrace{R_{\text{buy}}^{\ominus}(k)}_{\delta^k} + dN_t^{\lambda^{-}(\ell)}\right)\right)q,$$

and for the cost function at the ask:

$$dJ^{u}(Q^{a},Q^{b}) = \left[\frac{q}{Q^{a}}p^{\mathrm{buy}}(Q^{a}) + \left(1 - \frac{q}{Q^{a}}\right)J^{u}(Q^{a} - q,Q^{b})\right](dN_{t}^{\lambda\mathrm{buy}(k)}\underbrace{\mathcal{R}_{\mathrm{buy}}^{\ominus}(k)}_{\delta^{k}} + dN_{t}^{\lambda^{-}(\ell)}) - c_{a}q\,dt.$$

Again, with T large enough, $u(\mathcal{Q}^a, \mathcal{Q}^b) = I\!\!E \int_{t=0}^T J(Q_t^a, Q_t^b) dt$ given $Q_0^a = \mathcal{Q}^a, Q_0^b = \mathcal{Q}^b$, and

$$\mathcal{U}(\mathcal{Q}^{a},\mathcal{Q}^{b}) := \max_{\delta^{i} \in \{0,1\}} \delta^{i} u(\mathcal{Q}^{a},\mathcal{Q}^{b}) + (1-\delta^{i}) p^{\mathrm{sell}}(\mathcal{Q}^{b})$$

The control δ^i is thus the result of a **backward** process.

One of the Results: Four regions

Four mixes of LC and/or LP agents:



Sellers and buyers are Liquidity Providers

 $R^{++} = \{(x, y), R^{\oplus}_{s}(x, y) = R^{\oplus}_{b}(x, y) = 1\},\$

Sellers and buyers are Liquidity Consumers

$$R^{--} = \{(x, y), R_s^{\ominus}(x, y) = R_b^{\ominus}(x, y) = 1\},\$$

Sellers provide liquidity and buyers consume it

$$R^{+-} = \{(x, y), R_s^{\oplus}(x, y) = R_b^{\ominus}(x, y) = 1\},\$$

Sellers consume liquidity and buyers provide it

$$R^{-+} = \{(x, y), R_s^{\ominus}(x, y) = R_b^{\oplus}(x, y) = 1\}.$$

MFG For Oderbook Dynamics

Outcomes of the modelling



The Liquidity Game

- It is possible to write properly the value of one limit order in a book (and to obtain its stationarized value);
- In the MFG model with one type of agent only, liquidity imbalance can be stable, it is in contradiction with some empirical findings;
- But if we mix trading speeds, no imbalance is stable anymore (see the paper).

The results are compatible with empirical studies ([Gareche et al., 2013], [Huang et al., 2015]; see [Bouchaud et al., 2018] for more details).

This model focuses on the **Liquidity Game**: traders compete for liquidity, while there is no (exogenous, i.e. fundamental) reasons for a price change. In reality, such a situation alternates with **price focussed sequences**; see [Huang et al., 2015].

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MFG of Controls and An Application To Trade Crowding

Motivation

Once they took the decision to update their portfolios, asset managers (think about a pension fund) have to go on markets to buy or sell the corresponding number of shares or contracts.

- the common (good) practice is that all the portfolio managers send their instructions to the dealing desk of their company;
- ► The job of this team is to "time" the execution of these large orders:
- if they go too fast, the trading pressure will more the prices an unfavourable way (market impact),
- if they go too slow, they will miss the "good opportunity" to buy or sell.
- $\rightarrow\,$ hence they main goal is to adjust their trading speed in real time.
- That for, they build and use trading algorithms (theirs or the ones of their brokers) to interact with market an optimized way.
- Each algorithm is parametrized thanks to standardized cost functions.

Trading algorithms implementations are inspired by the same formalism we will use here (books are widely available: [Guo et al., 2016], [Cartea et al., 2015], [Guéant, 2016] and [L. and Laruelle., 2018a]).

MFG of Controls and An Application To Trade Crowding

A continuum of agents trading optimally "à la Cartea-Jaimungal".

 $dS_t = \alpha \mu_t \, dt + \sigma \, dW_t.$

$$dQ_t^a = \nu_t^a \, dt$$

now for a seller, $Q_0^a > 0$ (the associated control ν^a will be mostly negative) and the wealth suffers from linear trading costs driven by κ (or *temporary*, or *immediate market impact*):

$$dX_t^a = -\nu_t^a (S_t + \kappa \cdot \nu_t^a) dt.$$

Same equations as for the standard framework, except the trend is made of the permanent impact of all agents:

$$\mu_t = \int_{a\in\mathfrak{A}} \nu_t^a \, dm(a),$$

where f(a) is the density of the agents in a feature space \mathfrak{A} .

From Cost Function to HJB

The cost function of investor *a* selling from t = 0 and *T* is similar to the ones used in [Cartea et al., 2015]: the terminal inventory is penalized and a quadratic running cost is subtracted:

$$V^a_t := \sup_{
u} I\!\!E\left(X^a_T + Q^a_T(S_T - A^a \cdot Q^a_T) - \phi^a \int_{s=t}^T (Q^a_s)^2 ds \Big| \mathcal{F}_t
ight)$$

Here we took T common to all investors, i.e. the end of the trading day.

Our framework is then

- ► Each agent a has an initial quantity Q₀^a to buy (Q₀^a < 0) or to sell (Q₀^a > 0) we can even have purely opportunistic agents (Q₀^a = 0).
- They all start at the open of the trading session t = 0 and end at the close t = T.
- Each of them maximizes the value of his trades for the day: cash + penalized remaning quantity (by A^a) cost of risk (with his own risk aversion ϕ^a).

HJB For One Player (Backward Value Function)

The associated Hamilton-Jacobi-Bellman is

$$0 = \partial_t V^a - \phi^a q^2 + \frac{1}{2} \sigma^2 \partial_S^2 V^a + \alpha \mu \partial_S V^a + \sup_{\nu} \left\{ \nu \partial_Q V^a - \nu (s + \kappa \nu) \partial_X V^a \right\},$$

with the terminal condition $V^a(T, x, s, q; \mu) = x + q(s - A^a q)$. The usual solution: Following the Cartea and Jaimungal's approach, we will use the following ersatz: $V^a = x + qs + v^a(t, q; \mu)$. Thus the HJB on v is

$$-\alpha\mu q = \partial_t v^a - \phi^a q^2 + \sup_{\nu} \left\{ \nu \partial_Q v^a - \kappa \nu^2 \right\},$$

with the teminal condition $v^a(T, q; \mu) = -A^a q^2$.

The associated optimal feedback / control is straightforward to find:

(2)
$$\nu^{a}(t,q) = \frac{\partial_{O} \nu^{a}(t,q)}{2\kappa}$$

 \Rightarrow We know that if we have the value function of an agent v, we can deduce the associated optimal control.

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Transport of the Mass of the Players (Forward)

Distribution of agents is mainly defined by the joint distribution m(t, dq, da) of

- the inventory Q_t^a , with known initial values.
- the preferences of the agent: the risk aversion ϕ^a , and the terminal penalization A^a .

The net trading flow μ driving the trend of the public price at time *t* reads:

$$\mu_t = \int_{(q,a)} \nu_t^a(q) m(t, dq, da) = \int_{q,a} \frac{\partial_Q v^a(t,q)}{2\kappa} m(t, dq, da).$$

 $\Rightarrow v^a$ is an implicit function of μ (look at the HJB), meaning we will have a fixed point problem to solve in μ . By the dynamics of Q_t^a , the transport of the measure m(t, dq, da) has to follow (continuity equation)

$$\partial_t m + \partial_q \left(m \frac{\partial_Q v^a}{2\kappa} \right) = 0$$
 with initial condition $m_0 = m_0(dq, da)$.

Obtaining The Backward-Forward Dynamics

Now we can have side to side:

- the HJB (backward) PDE where we plug the value of μ ;
- ▶ the (Forward) transport of the mass of agents *m*, driven by the aggregation of their instantaneous decisions.

$$\begin{cases} -\alpha q \underbrace{\int_{(q',a')} \frac{\partial_Q v^{a'}(t,q')}{2\kappa} m(t,dq',da')}_{\text{aggregate of all agents}} = \underbrace{\partial_t v^a - \phi^a q^2 + \frac{(\partial_Q v^a)^2}{4\kappa}}_{\text{optimal for one agent}} \\ \partial_t m + \partial_q \left(m \frac{\partial_Q v^a}{2\kappa}\right) = 0 \end{cases}$$

Under boundary (resp. initial and terminal) conditions:

Explicit Solution For a Special Case

Same preferences for all agents: $\phi^{a} \equiv \phi, \, \mathbf{A}^{a} \simeq \mathbf{A}$

We will need a notation for the aggregated (i.e. net) position of all agents $E(t) = \int_{q} q m(t, dq)$. Then we can write:

$$\begin{array}{lcl} E'(t) & = & \int_{q} q \partial_{t} m(t, dq) & \leftarrow & \text{definition} \\ & = & - \int_{q} q \partial_{q} \left(m(t, q) \frac{\partial_{Q} v(t, q)}{2\kappa} \right) dq & \leftarrow & \text{forward dynamics (transport} \\ & = & \int_{q} \frac{\partial_{Q} v(t, q)}{2\kappa} m(t, dq) & \leftarrow & \text{integration by parts.} \end{array}$$

Moreover, v(t,q) can be expressed as a quadratic function of q: $v(t,q) = h_0(t) + q h_1(t) - q^2 \frac{h_2(t)}{2}$, leading to:

$$E'(t) = \int_q m(t,q) \left(\frac{h_1(t)}{2\kappa} - \frac{h_2(t)}{2\kappa}q\right) dq = \frac{h_1(t)}{2\kappa} - \frac{h_2(t)}{2\kappa}E(t).$$

In a more compact form:

$$2\kappa E'(t) = h_1(t) - E(t) \cdot h_2(t)$$

Dynamics For Identical Preferences

You inject h_0 , h_1 and h_2 in your control PDE tool and you collect all the obtained ODEs:

(3a)
$$(4\kappa\phi = -2\kappa h'_2(t) + (h_2(t))^2,$$

(3b)
$$\alpha h_2(t)E(t) = 2\kappa h'_1(t) + h_1(t)(\alpha - h_2(t))$$

(3c)
$$-(h_1(t))^2 = 4\kappa h'_0(t),$$

(3d)
$$2\kappa E'(t) = h_1(t) - h_2(t)E(t).$$

with the boundary conditions $h_0(T) = h_1(T) = 0$, $h_2(T) = 2A$, $E(0) = E_0$, where $E_0 = \int_q qm_0(q)dq$ is the net initial inventory of market participants (i.e. the expectation of the initial density *m*).

The Main Equation For Identical Preferences

The previous system of ordinary differential equations implies

(4)
$$0 = 2\kappa E''(t) + \alpha E'(t) - 2\phi E(t)$$

with boundary conditions $E(0) = E_0$ and $\kappa E'(T) + AE(T) = 0$.

Solving The Mean Field

Closed form for the net inventory dynamics E(t)

For any $\alpha \in \mathbb{R}$, the problem (4) has a unique solution *E*, given by

$$E(t) = E_0 a(\exp\{r_+t\} - \exp\{r_-t\}) + E_0 \exp\{r_-t\}$$

where a is given by

$$a = \frac{(\alpha/4 + \kappa\theta - A)\exp\{-\theta T\}}{-\frac{\alpha}{2}\mathrm{sh}\{\theta T\} + 2\kappa\theta\mathrm{ch}\{\theta T\} + 2A\mathrm{sh}\{\theta T\}},$$

the denominator being positive and the constants r^{\pm}_{α} and heta being given by

$$r_{\pm} := -rac{lpha}{4\kappa} \pm heta, \qquad heta := rac{1}{\kappa} \sqrt{\kappa \phi + rac{lpha^2}{16}}$$

Solving the Control

Solving $h_2(t)$

 h_2 solves the following backward ordinary differential equation (3a): $0 = 2\kappa \cdot h'_2(t) + 4\kappa \cdot \phi - (h_2(t))^2$ under $h_2(T) = 2A$. It is easy to check the solution is

$$h_2(t) = 2\sqrt{\kappa\phi}\frac{1+c_2e^{rt}}{1-c_2e^{rt}},$$

where $r = 2\sqrt{\phi/\kappa}$ and c_2 solves the terminal condition. Hence

$$c_2 = -\frac{1-A/\sqrt{\kappa\phi}}{1+A/\sqrt{\kappa\phi}} \cdot e^{-rT}$$

Keep in mind the optimal control is

$$u^* = rac{\partial_Q v(t,q)}{2\kappa} = rac{h_1(t)}{2\kappa} - q \cdot rac{h_2(t)}{2\kappa},$$

Solving $h_1(t)$

The affine component of the control can be easily deduced from $h_2(t)$ and E(t):

 $h_1(t) = 2\kappa \cdot E'(t) + h_2(t) \cdot E(t).$

Qualitative Meaning of All This

(Back To Identical Preferences To Ease The Writing)



- ▶ The first term embeds the dependence to the mean field : $h_1(t) = 2\kappa \cdot E'(t) + h_2(t) \cdot E(t)$.
- \Rightarrow locally you adapt your behaviour to the mean field via h_1 ,
- ightarrow then (you changed your inventory), you slowly (re)adapt to be ready for boundary conditions / costs.

The Standard Case



Conclusion

A topic of paramount importance for financial participants is liquidity .

- ► In a lot of cases, liquidity can be seen as a mean field in the sense of the MFG
- Financial mathematics traditionally uses stochastic control to exhibit optimal strategies for an isolated market participant in front of a "background noise".
- Mean Field Games naturally extends this kind of frameworks to take into account the dynamics of liquidity driven by other players trying to be optimal too.
- The obtained results ar of interest for
- intermediaries (banks, brokers, insurance companies, etc)
- asset managers
- regulators.
- This kind of MFG can be considered as a way to obtain robust control
- It is possible to embed partial information and learning.

Thank You For Your Attention



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References I

	Almgren, R. F. and Chriss, N. (2000).
	Optimal execution of portfolio transactions.
	Journal of Risk, 3(2):5–39.
	Azencott, R., Beri, A., Gadhvan, Y., Joseph, N., Lehalle, CA., and Rowley, M. (2014).
	Bealtime market microstructure analysis: online Transaction Cost Analysis.
	Quantitative Finance, pages 0–19.
	Bensoussan, A., Sung, K. C. J., Yam, S. C. P., and Yung, S. P. (2016).
	Linear-quadratic mean field games.
	Journal of Optimization Theory and Applications, 169(2):496–529.
	Bouchard B. Dang N - M. and Leballe C - A (2011)
	Optimal control of trading algorithms: a general impulse control approach.
_	SIAM J. FINANCIAI MAINEMAIICS, 2(1):404–438.
	Bouchaud, JP., Bonart, J., Donier, J., and Gould, M. (forthcoming (2018)).
	Trades, Quotes and Prices: Financial Markets Under the Microscope.
	Cambridge University Press.

References II

CFM

Cardaliaguet, P. and Lehalle, CA. (2017).
Mean Field Game of Controls and An Application To Trade Crowding.
Mathematics and Financial Economics, pages 1–29.
Carmona, R., Fouque, JP., and Sun, LH. (2013).
Mean Field Games and Systemic Risk.
Social Science Research Network Working Paper Series.
Cartea, A., Jaimungal, S., and Penalva, J. (2015).
Algorithmic and High-Frequency Trading (Mathematics, Finance and Risk).
Cambridge University Press, 1 edition.
Firoozi, D. and Caines, P. E. (2016).
Mean Field Game epsilon-Nash equilibria for partially observed optimal execution problems in finance.
In 2016 IEEE 55th Conference on Decision and Control (CDC), pages 268–275.
Foucault, T. and Menkveld, A. J. (2008).
Competition for Order Flow and Smart Order Routing Systems.

The Journal of Finance, 63(1):119–158.

References III

CFM



Mean-Field Game Strategies for a Major-Minor Agent Optimal Execution Problem. Social Science Research Network Working Paper Series.

References IV

CFM

Lachapelle, A., Lasry, J.-M., Lehalle, C.-A., and Lions, P.-L. (2016). Efficiency of the Price Formation Process in Presence of High Frequency Participants: a Mean Field Game analysis. Mathematics and Financial Economics, 10(3):223-262. Laruelle, S., Lehalle, C.-A., and Pagès, G. (2013). Optimal posting price of limit orders: learning by trading. Mathematics and Financial Economics, 7(3):359-403. Lehalle, C.-A., Laruelle, S., Burgot, R., Pelin, S., and Lasnier, M. (2018a). Market Microstructure in Practice. World Scientific publishing. Lehalle, C.-A. and Mouniid, O. (2016). Limit Order Strategic Placement with Adverse Selection Risk and the Role of Latency. Lehalle, C.-A., Mouniid, O., and Rosenbaum, M. (2018b). Optimal liquidity-based trading tactics. Lehalle, C.-A. and Neuman, E. (2017). Incorporating Signals into Optimal Trading.

Learning The Mean Field

Setup:

- rounds are days, the number of rounds n increases
- ► m(0, da, dq) stays (almost) the same every day

Agents:

- each agent has his own view on the aggregated trading speed $\mu
 ightarrow \mu^{a,n}$
- he implements the "optimal" strategy using $\mu^{a,n}$ in place of the true μ
- the effective aggregation of speeds is then m^{n+1} during this round
- each agent estimates it with an error
- ► and updates his view for the next round using a "memory" parameter:

$$\mu^{a,n+1}(t) := (1 - \pi^{a,n+1})\mu^{a,n}(t) + \pi^{a,n+1}(m^{n+1}(t) + \epsilon^{a,n+1}(t))$$

where ϵ is a bounded "observation error".

Dynamics of the Learning "Procedure"

When agent *a* "believes" that the mean filed will be $(\mu^{a,n})$ for today, she can compute hers optimal control:

$$h_1^{a,n}(t) = \alpha \int_t^T e^{\theta^a(t-s)} \mu^{a,n}(s) ds, \quad h_2^a = 2\kappa \theta^a.$$

Hence the effective mean field will in reality be the aggregation of such trading speeds (i.e. controls):

$$m^{n+1}(t) = \frac{1}{2\kappa} \left\{ \int_a h_1^{a,n}(t) \ \bar{m}_0(da) - \int_a h_2^{a,n} \ E^{a,n}(t) \ \bar{m}_0(da) \right\}.$$

That can be written to see the previous estimation of each agent:

$$m^{n+1}(t) = \frac{\alpha}{2\kappa} \left\{ \int_{t}^{T} ds \int_{a} (\mu^{a,n}(s)) e^{\theta^{a}(t-s)} \bar{m}_{0}(da) - \int_{0}^{t} d\tau \int_{\tau}^{T} ds \int_{a} (\mu^{a,n}(s)) e^{\theta^{a}(2\tau-t-s)} \theta^{a} \bar{m}_{0}(da) \right\}$$
$$- \int_{a} \theta^{a} e^{-\theta^{a}t} E_{0}^{a} \bar{m}_{0}(da).$$

CA Lehalle

Solving the Learning Game

Now each agent updates her belief for tomorrow:

$$\mu^{a,n+1}(t) := (1 - \pi^{a,n+1}) (\mu^{a,n}(t)) + \pi^{a,n+1} (m^{n+1}(t)) + \epsilon^{a,n+1}(t)).$$

If you express:

$$\sup_{a} \left| |\mu^{a,n} - \mu \right| |_{\infty} \leq \cdots .$$

and provided $\pi^{a,n}$ goes to zero fast enough (1/n), you obtain a law of large numbers:

Convergence of the incomplete information game

Within such a setup, this learning game converges towards its perfectly informed version:

$$\limsup_{a} \left| |\mu^{a,n} - \mu \right| |_{\infty} \leq C \left| |\epsilon^{a,\cdot} \right| |_{\infty}.$$

It is probably possible to obtain a CLT...